Journal of Mathematical Sciences: Advances and Applications
Volume 6, Number 1, 2010, Pages 137-148

APPLICATIONS OF FRACTIONAL DERIVATIVE OPERATOR TO CERTAIN SUBCLASSES OF MULTIVALENT FUNCTIONS

TARIQ O. SALIM

Department of Mathematics
Al-Azhar University-Gaza
P. O. Box 1277
Gaza
Palestine
e-mail: trsalim@yahoo.com

Abstract

In the present paper, some subclasses of multivalent functions defined in terms of a fractional derivative are studied. Coefficient inequalities and other interesting results are obtained.

1. Introduction

Let $A(p)$ be the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p}, \quad (p \in \mathbb{N}), \tag{1.1}$$

that are analytic and multivalent in the open unit disk $E = \{z : z \in \mathbb{C}, |z| < 1\}$.

The fractional calculus are defined as follows (e.g., [5, 6]).

2010 Mathematics Subject Classification: 30C45, 26A33.

Keywords and phrases: multivalent functions, fractional derivative operator.

Received September 24, 2010

Definition 1.1. The Riemann-Liouville fractional integral of order λ is defined for the function *f* by

$$D^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z (z-t)^{\lambda-1} f(t) dt, \quad (\lambda > 0), \tag{1.2}$$

where the function $f(z)$ is analytic in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ to be real when $z-t > 0$.

Definition 1.2. The Riemann-Liouville fractional derivative of order λ is defined for the function *f* by

$$D^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z (z-t)^{-\lambda} f(t) dt, \quad (0 \leq \lambda < 1), \tag{1.3}$$

where the function $f(z)$ is analytic in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z-t)^{-\lambda}$ is removed by requiring $\log(z-t)$ to be real when $z-t > 0$.

Definition 1.3. Under the hypothesis of Definition 1.2, the Riemann-Liouville fractional derivative of order $(q+\lambda)$ is defined for the function *f* by

$$D^{q+\lambda} f(z) = \frac{d^q}{dz^q} D^{\lambda} f(z), \quad (0 \leq \lambda < 1, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \tag{1.4}$$

Upon applying the fractional derivative to both sides of (1.1), we get

$$D^{q+\lambda} f(z) = \phi(p, q, \lambda) z^{p-q-\lambda} + \sum_{k=1}^{\infty} \phi(k+p, q, \lambda) a_{k+p} z^{k+p-q-\lambda}, \tag{1.5}$$

where $p > q$, $p \in \mathbb{N}$, $q \in \mathbb{N}_0$, $0 \leq \lambda < 1$, and

$$\phi(p, q, \lambda) = \frac{\Gamma(p+1)}{\Gamma(p-q-\lambda+1)}. \tag{1.6}$$

A function $f(z) \in A(p)$ is said to be *p-valent* λ*- starlike function* of order $\alpha,$ if it satisfies the inequality

$$\operatorname{Re}\left(\frac{zD^{\lambda+1}f(z)}{D^{\lambda}f(z)}\right) > \alpha, \quad (z \in E), \tag{1.7}$$

where $0 \leq \alpha < p, 0 \leq \lambda < 1,$ and $p \in \mathbb{N}.$ We denote by $S^{\alpha}(p, \lambda)$ the class of p-valent λ- starlike functions of order $\alpha.$

A function $f(z) \in A(p)$ is said to be *p-valent* λ*- convex function* of order $\alpha,$ if it satisfies the inequality

$$\operatorname{Re}\left(1 + \frac{zD^{\lambda+2}f(z)}{D^{\lambda+1}f(z)}\right) > \alpha, \quad (z \in E), \tag{1.8}$$

where $0 \leq \alpha < p, 0 \leq \lambda < 1,$ and $p \in \mathbb{N}.$ We denote by $C^{\alpha}(p, \lambda)$ the class of p-valent λ- convex functions of order $\alpha.$

Note that when $\lambda = 0,$ then $S^{\alpha}(p) \equiv S^{\alpha}(p, 0)$ and $C^{\alpha}(p) \equiv C^{\alpha}(p, 0)$ are the well-known classes of p-valent starlike functions of order α and p-valent convex functions of order $\alpha,$ see, e.g., [1, 2, 7].

Now, let us define the following subclass of multivalent analytic functions as follows:

Definition 1.4. The class $T^{\alpha}(p, q, \lambda)$ consists of functions $f(z) \in A(p)$ satisfying the inequality

$$\left|\frac{zD^{q+\lambda+1}f(z)}{D^{q+\lambda}f(z)} - p + q\right| < p - \alpha, \tag{1.9}$$

where $p > q,$ $p \in \mathbb{N},$ $q \in \mathbb{N}_0,$ $0 \leq \lambda < 1,$ and $0 \leq \alpha < p.$

Note that on setting $q = 0$ and $q = 1$ in Definition 1.4, we easily get the classes $S^{\alpha}(p, \lambda)$ and $C^{\alpha}(p, \lambda),$ respectively. In other words, $T^{\alpha}(p, 0, \lambda) = S^{\alpha}(p, \lambda)$ and $T^{\alpha}(p, 1, \lambda) = C^{\alpha}(p, \lambda).$

The purpose of the present investigation is to focus on the functions $f(z)$ belonging to the class $T^{\alpha}(p, q, \lambda)$ defined by (1.9) and to obtain coefficient inequalities and some other interesting results.

It is useful here to recall Jack's lemma [4], which is needed in the present work.

Lemma 1.5. *Let* $\omega(z)$ *be a non-constant analytic function in E with* $\omega(0) = 0.$ *If* $|\omega(z)|$ *attains its maximum value on the circle* $|z| = r < 1$ *at a point* $z_0,$ *then* $z_0\omega(z_0) = c\omega'(z_0),$ *where c is a real number satisfying* $c \geq 1.$

2. Results for the Class $T^{\alpha}(p, q, \lambda)$

We begin by proving the following theorem in which coefficient inequality for functions $f(z) \in A(p)$ belonging to the class $T^{\alpha}(p, q, \lambda)$ is established.

Theorem 2.1. *Let* $f(z) \in A(p)$ *satisfy*

$$\sum_{k=1}^{\infty} \frac{(p+1)_k}{(p+1-q-\lambda)_k} (k+p-\lambda-\alpha)|a_{k+p}| \leq (p-\alpha+\lambda), \tag{2.1}$$

where $p > q,\ p \in \mathbb{N},\ q \in \mathbb{N}_0,\ 0 \leq \lambda < 1,$ *and* $0 \leq \alpha < p.$ *Then* $f(z) \in T^{\alpha}(p, q, \lambda).$

Proof. Making use of (1.1) and (1.5), we get

$$\left|\frac{zD^{q+\lambda+1}f(z)}{D^{q+\lambda}f(z)} - p + q\right| = \left|\frac{zD^{q+\lambda+1}f(z) - (p-q)D^{q+\lambda}f(z)}{D^{q+\lambda}f(z)}\right|$$

$$= \left|\frac{[\phi(p, q+1, \lambda) - (p-q)\phi(p, q, \lambda)]z^{p-q-\lambda} - \sum_{k=1}^{\infty}[\phi(k+p, q+1, \lambda) - (p-q)\phi(k+p, q, \lambda)]a_{k+p}z^{k+p-q-\lambda}}{\phi(p, q, \lambda)z^{p-q-\lambda} + \sum_{k=1}^{\infty}\phi(k+p, q, \lambda)a_{k+p}z^{k+p-q-\lambda}}\right|$$

$$\leq \frac{|\phi(p, q+1, \lambda) - (p-q)\phi(p, q, \lambda)||z|^{p-q-\lambda} + \sum_{k=1}^{\infty} |\phi(k+p, q+1, \lambda) - (p-q)\phi(k+p, q, \lambda)||a_{k+p}||z|^{k+p-q-\lambda}}{\phi(p, q, \lambda)|z|^{p-q-\lambda} - \sum_{k=1}^{\infty} \phi(k+p, q, \lambda)|a_{k+p}||z|^{k+p-q-\lambda}}$$

$$< \frac{|\phi(p, q+1, \lambda) - (p-q)\phi(p, q, \lambda)| + \sum_{k=1}^{\infty} |\phi(k+p, q+1, \lambda) - (p-q)\phi(k+p, q, \lambda)||a_{k+p}|}{\phi(p, q, \lambda) - \sum_{k=1}^{\infty} \phi(k+p, q, \lambda)|a_{k+p}|}. \tag{2.2}$$

This expression is bounded by $(p - \alpha)$, if

$$\sum_{k=1}^{\infty} [\phi(k+p, q+1, \lambda) + (q-\alpha)\phi(k+p, q, \lambda)]|a_{k+p}|$$

$$\leq (p-\alpha)\phi(p, q, \lambda) - |\phi(p, q+1, \lambda) - (p-q)\phi(p, q, \lambda)|$$

$$\leq (2p - q - \alpha)\phi(p, q, \lambda) - \phi(p, q+1, \lambda). \tag{2.3}$$

Thus, by applying (1.6) and making use of the Pochhammer symbol $\frac{\Gamma(a+n)}{\Gamma(a)} = (a)_n$, we easily get the (2.1) as a condition for $f(z)$ to belong to the class $T^{\alpha}(p, q, \lambda)$. □

Theorem 2.2. *If* $f(z) \in A(p)$ *satisfies the inequality*

$$\left| \frac{1 + \frac{zD^{q+\lambda+2}f(z)}{D^{q+\lambda+1}f(z)} - p + q + \lambda}{\frac{zD^{q+\lambda+1}f(z)}{D^{q+\lambda}f(z)} - p + q + \lambda} - 1 \right| < \frac{1}{2p - q - \alpha - 2\lambda}, \tag{2.4}$$

where $p > q$, $p \in \mathbb{N}$, $q \in \mathbb{N}_0$, $0 \leq \lambda < 1$, *and* $0 \leq \alpha < p - \lambda$. *Then* $f(z) \in T^{\alpha}(p, q, \lambda)$.

Proof. Define the function $\omega(z)$ by

$$\omega(z) = \frac{1}{p - \alpha - \lambda}\left[\frac{zD^{q+\lambda+1}f(z)}{D^{q+\lambda}f(z)} - p + q + \lambda\right]. \tag{2.5}$$

It can be easily verified that $\omega(z)$ satisfies the hypothesis of Lemma 1.5.

By logarithmic differentiation of (2.5), we can easily obtain

$$1 + \frac{zD^{q+\lambda+2}f(z)}{D^{q+\lambda+1}f(z)} - p + q + \lambda$$

$$= (p - \alpha - \lambda)\left[1 + \frac{z\omega'(z)}{\omega(z)} \cdot \frac{1}{p - q - \lambda + (p - \alpha - \lambda)\omega(z)}\right]. \tag{2.6}$$

Now, let

$$G(z) = \frac{1 + \frac{zD^{q+\lambda+2}f(z)}{D^{q+\lambda+1}f(z)} - p + q + \lambda}{\frac{zD^{q+\lambda+1}f(z)}{D^{q+\lambda}f(z)} - p + q + \lambda} - 1$$

$$= \frac{z\omega'(z)}{\omega(z)} \cdot \frac{1}{p - q - \lambda + (p - \alpha - \lambda)\omega(z)}. \tag{2.7}$$

Therefore, from Lemma 1.5 and (2.7), we get

$$|G(z_0)| = \left|\frac{z\omega'(z_0)}{\omega(z_0)} \cdot \frac{1}{p - q - \lambda + (p - \alpha - \lambda)\omega(z_0)}\right|$$

$$\geq \frac{c}{2p - q - \alpha - 2\lambda}, \quad c \geq 1,$$

which contradicts the inequality (2.4). Hence, we must have $|\omega(z)| < 1$ for all $z \in E$.

So, we have

$$\frac{1}{p - \alpha - \lambda}\left\{\left|\frac{zD^{q+\lambda+1}f(z)}{D^{q+\lambda}f(z)} - p + q\right| - \lambda\right\} < |\omega(z)| < 1,$$

which directly yields that $f(z) \in T^{\alpha}(p, q, \lambda)$. □

By choosing suitable values of the parameters $p, q,$ and $\lambda,$ one can obtain several special cases concerning the coefficient inequalities and other interesting results for various subclasses of multivalent and univalent functions. Below, some of these results are contained in the following corollaries.

By setting $q = 0$ and $q = 1$, respectively, in Theorem 2.1, we get:

Corollary 2.3. *Let* $f(z) \in A(p)$ *satisfy*

$$\sum_{k=1}^{\infty} \frac{(p+1)_k}{(p+1-\lambda)_k} (k+p-\lambda-\alpha)|a_{k+p}| \leq (p-\alpha+\lambda), \tag{2.8}$$

where $p \in \mathbb{N}$, $0 \leq \lambda < 1$, *and* $0 \leq \alpha < p$. *Then* $f(z) \in S^{\alpha}(p, \lambda)$.

Corollary 2.4. *Let* $f(z) \in A(p)$ *satisfy*

$$\sum_{k=1}^{\infty} \frac{(p+1)_k}{(p-\lambda)_k} (k+p-\lambda-\alpha)|a_{k+p}| \leq (p-\alpha+\lambda), \tag{2.9}$$

where $p \in \mathbb{N}$, $0 \leq \lambda < 1$, *and* $0 \leq \alpha < p$. *Then* $f(z) \in C^{\alpha}(p, \lambda)$.

Moreover, if we set $\lambda = 0$ in Corollaries 2.3 and 2.4, we get the coefficient inequalities for the subclasses $S^{\alpha}(p)$ and $C^{\alpha}(p)$ of multivalent starlike and convex functions of order α, respectively.

Now, setting $q = 0$ and $q = 1$, respectively, in Theorem 2.2, we obtain:

Corollary 2.5. *If* $f(z) \in A(p)$ *satisfies*

$$\left| \frac{1 + \dfrac{zD^{\lambda+2}f(z)}{D^{\lambda+1}f(z)} - p + \lambda}{\dfrac{zD^{\lambda+1}f(z)}{D^{\lambda}f(z)} - p + \lambda} - 1 \right| < \frac{1}{2p - \alpha - 2\lambda}, \tag{2.10}$$

where $p \in \mathbb{N}$, $0 \leq \lambda < 1$, *and* $0 \leq \alpha < p$. *Then* $f(z) \in S^{\alpha}(p, \lambda)$.

Corollary 2.6. *If* $f(z) \in A(p)$ *satisfies*

$$\left| \frac{2 + \dfrac{zD^{\lambda+2}f(z)}{D^{\lambda+1}f(z)} - p + \lambda}{\dfrac{zD^{\lambda+1}f(z)}{D^{\lambda}f(z)} - p + \lambda} - 1 \right| < \frac{1}{2p - 1 - \alpha - 2\lambda}, \tag{2.11}$$

where $p \in \mathbb{N}$, $0 \leq \lambda < 1$, *and* $0 \leq \alpha < p$. *Then* $f(z) \in C^{\alpha}(p, \lambda)$.

Moreover, let $\lambda = 0$ in Corollary 2.5, we get:

Corollary 2.7. *If* $f(z) \in A(p)$, *satisfies*

$$\left| \frac{1 + \frac{zf''(z)}{f'(z)} - p}{\frac{zf'(z)}{f(z)} - p} - 1 \right| < \frac{1}{2p - \alpha}, \tag{2.12}$$

where $p \in \mathbb{N}$ *and* $0 \leq \alpha < p$. *Then* $f(z) \in S^{\alpha}(p)$.

3. More Results

In their paper, Irmak and Cho [3] studied the multivalent analytic functions in the open unit disk by using integer order differential operator. The results obtained are believed to be useful in geometric function theory. In this section, it is intended to extend these results by using a fractional order differential operator.

Theorem 3.1. *Let* $f(z) \in A(p)$, *then*

$$\operatorname{Re}\left(\frac{zD^{q+\lambda+1}f(z)}{D^{q+\lambda}f(z)} \right) < p - q - \lambda$$

$$\Rightarrow |D^{q+\lambda}f(z)| < \phi(p, q, \lambda)|z|^{p-q-\lambda-1}, \tag{3.1}$$

where $p > q$, $p \in \mathbb{N}$, $q \in \mathbb{N}_0$, $0 \leq \lambda < 1$, *and* $\phi(p, q, \lambda)$ *is given by* (1.6).

Proof. Let $f(z) \in A(p)$ and $\omega(z)$ is defined as

$$D^{q+\lambda}f(z) = \phi(p, q, \lambda)z^{p-q-\lambda-1}\left(z + \sum_{k=1}^{\infty} \frac{\phi(k+p, q, \lambda)}{\phi(p, q, \lambda)} a_{k+p} z^{k+1}\right)$$

$$= \phi(p, q, \lambda)z^{p-q-\lambda-1}\omega(z). \tag{3.2}$$

Differentiating (3.2) yields

$$zD^{q+\lambda}f(z) = \phi(p, q, \lambda)\omega(z)\left(p - q - \lambda - 1 + \frac{z\omega'(z)}{\omega(z)}\right)z^{p-q-\lambda-1}. \tag{3.3}$$

So, in view of (3.2) and (3.3), we have

$$\frac{zD^{q+\lambda+1}f(z)}{D^{q+\lambda}f(z)} = p - q - \lambda - 1 + \frac{z\omega'(z)}{\omega(z)}, \quad (\omega(z) \neq 0). \tag{3.4}$$

It is clear that $\omega(z)$ satisfies the hypothesis of Lemma 1.5. We claim that $|\omega(z)| < 1$. Indeed, if not, there exists $z_0 \in E$ such that $\max_{|z| \leq |z_0|} |\omega(z)| = |\omega(z_0)| = 1$. Since, we have $z_0 \omega'(z_0) = c\omega(z_0)$, $(c \geq 1)$, from Lemma 1.5. Thus with $z = z_0$, we have from (3.4) that

$$\operatorname{Re}\left(\frac{z_0 D^{q+\lambda+1}f(z_0)}{D^{q+\lambda}f(z_0)}\right) = p - q - \lambda - 1 + \operatorname{Re}\left(\frac{z_0\omega'(z_0)}{\omega(z_0)}\right)$$

$$\geq p - q - \lambda, \tag{3.5}$$

which contradicts the condition in (3.1). Therefore (3.2) yields

$$\left|\frac{D^{q+\lambda}f(z)}{z^{p-q-\lambda-1}}\right| = \phi(p, q, \lambda)|\omega(z)| \leq \phi(p, q, \lambda),$$

which directly implies the result (3.1). □

Theorem 3.2. *Let* $f(z) \in A(n)$ *and* $g(z) \in A(m)$ *with* $p = n - m$, $(p, n, m \in \mathbb{N})$ *and suppose that*

$$\operatorname{Re}\left(\frac{D^{q+\lambda}g(z)}{zD^{q+\lambda+1}g(z)}\right) > \beta, \quad (z \in E, q \in \mathbb{N}_0, 0 \leq \lambda < 1, \beta \geq 0). \tag{3.6}$$

If the inequality

$$\left|\frac{D^{q+\lambda}g(z)}{D^{q+\lambda}f(z)}\left(\frac{D^{q+\lambda+1}f(z)}{D^{q+\lambda+1}g(z)} - \frac{\phi(n, q, \lambda)}{\phi(m, q, \lambda)}\right)z^p\right| < \beta\left(p + \frac{1}{2}\right) - \frac{1}{2}, \tag{3.7}$$

holds. Then

$$\left|\frac{\phi(m, q, \lambda)}{\phi(n, q, \lambda)} \cdot \frac{D^{q+\lambda}f(z)}{D^{q+\lambda}g(z)} - z^p\right| < |z|^p, \tag{3.8}$$

for $p > q$, $p \in \mathbb{N}$, $q \in \mathbb{N}_0$, *and* $0 \leq \lambda < 1$.

Proof. In view of (1.1), (1.5), and (1.6), it is easily seen that

$$\frac{\phi(m, q, \lambda)}{\phi(n, q, \lambda)} \cdot \frac{D^{q+\lambda} f(z)}{D^{q+\lambda} g(z)} = z^p \left(1 + \sum_{k=1}^{\infty} c_{k+p} z^k\right) \in A(p) = A(m-n). \quad (3.9)$$

Now define the function $\omega(z)$ by

$$\frac{\phi(m, q, \lambda)}{\phi(n, q, \lambda)} \cdot \frac{D^{q+\lambda} f(z)}{D^{q+\lambda} g(z)} = z^p [1 + \omega(z)]. \quad (3.10)$$

Differentiating (3.10), then we get

$$\frac{\phi(m, q, \lambda)}{\phi(n, q, \lambda)} \cdot \frac{D^{q+\lambda+1} f(z)}{z^p D^{q+\lambda+1} g(z)} - 1$$

$$= \omega(z) + [z\omega'(z) + p(1 + \omega(z))] \frac{D^{q+\lambda} g(z)}{z D^{q+\lambda+1} g(z)}. \quad (3.11)$$

Now define the function $F(z)$ by

$$F(z) = \frac{\left(\frac{D^{q+\lambda+1} f(z)}{z^p D^{q+\lambda+1} g(z)} - \frac{\phi(n, q, \lambda)}{\phi(m, q, \lambda)}\right)}{\left(\frac{D^{q+\lambda} f(z)}{z^p D^{q+\lambda} g(z)}\right)}. \quad (3.12)$$

Then, in view of (3.10), we have

$$F(z) = \frac{\omega(z)}{1 + \omega(z)} + \left[p + \frac{z\omega'(z)}{\omega(z)}\right] \frac{D^{q+\lambda} g(z)}{z D^{q+\lambda+1} g(z)}. \quad (3.13)$$

Now, as in the proof of Theorem 3.1, we claim that $|\omega(z)| < 1$, otherwise,

$$|F(z_0)| = \left| \frac{\omega(z_0)}{1 + \omega(z_0)} + \left[p + \frac{z\omega'(z_0)}{\omega(z_0)}\right] \frac{D^{q+\lambda} g(z_0)}{z_0 D^{q+\lambda+1} g(z_0)} \right|$$

$$\geq \left| \left[p + \frac{z\omega'(z_0)}{\omega(z_0)}\right] \frac{D^{q+\lambda} g(z_0)}{z_0 D^{q+\lambda+1} g(z_0)} \right| - \left| \frac{\omega(z_0)}{1 + \omega(z_0)} \right|$$

$$\geq \operatorname{Re}\left(p + \frac{z\omega'(z_0)}{\omega(z_0)}\right)\operatorname{Re}\left(\frac{D^{q+\lambda}g(z_0)}{z_0 D^{q+\lambda+1}g(z_0)}\right) - \operatorname{Re}\left(\frac{\omega(z_0)}{1+\omega(z_0)}\right)$$

$$\geq \beta(p + \frac{1}{2}) - \frac{1}{2}, \tag{3.14}$$

which contradicts (3.7). Hence $|\omega(z)| < 1$ for all $z \in E$, and (3.10) evidently yields the inequality (3.8). □

Theorem 3.3. *Let* $f(z) \in A(p)$, *then*

$$\left| D^{q+\lambda} f(z)\left(\frac{zD^{q+\lambda+1}f(z)}{D^{q+\lambda}f(z)} - p + q + \lambda\right)\right| < |z|^{p-q-\lambda}$$

$$\Rightarrow |D^{q+\lambda}f(z) - \phi(p, q, \lambda)z^{p-q-\lambda}| < |z|^{p-q-\lambda}, \tag{3.15}$$

where $p > q$, $p \in \mathbb{N}$, $q \in \mathbb{N}_0$, $0 \leq \lambda < 1$, *and* $\phi(p, q, \lambda)$ *is given by* (1.6).

Proof. Making use of (1.5) for $f(z) \in A(p)$, we get

$$\frac{D^{q+\lambda}f(z)}{z^{p-q-\lambda}} - \phi(p, q, \lambda) = \sum_{k=1}^{\infty}\phi(k + p, q, \lambda)a_{k+p}z^k = \omega(z). \tag{3.16}$$

Differentiating (3.16) implies

$$\frac{D^{q+\lambda}f(z)}{z^{p-q-\lambda}} = (p - q - \lambda)[\phi(p, q, \lambda) + \omega(z)] + z\omega'(z). \tag{3.17}$$

So, we have

$$z\omega'(z) = \frac{D^{q+\lambda}f(z)}{z^{p-q-\lambda}}\left(\frac{zD^{q+\lambda+1}f(z)}{D^{q+\lambda}f(z)} - p + q + \lambda\right). \tag{3.18}$$

Now, applying Lemma 1.5, if $z = z_0$, we obtain $|z_0\omega'(z_0)| = c|\omega(z_0)| = c \geq 1$, which contradicts the condition in (3.15). So, we have $|\omega(z)| < 1$ for all $z \in E$, which completes the proof of the theorem. □

Note that the results obtained by Irmak and Cho [3] are special cases of the theorems above, which can be deduced by setting $\lambda = 0$.

References

[1] P. K. Banerji and G. M. Shenan, Certain generalized subclasses of analytic and multivalent functions in terms of fractional calculus, Proc. Nat. Acad. Sci. India. 71(A) (2001), 321-338.

[2] P. L. Duren, Univalent Functions, Springer-Verlag, New York, 1983.

[3] H. Irmak and N. E. Cho, A differential operator and its applications to certain multivalently analytic functions, Hacettepe J. Math. Stat. 36(1) (2007), 1-6.

[4] I. S. Jack, Functions starlike and convex of order α, J. London Math. Soc. 3 (1971), 469-474.

[5] S. Owa, On the distortion theorems-I, Kyungpook Math. J. 18(1) (1978), 53-59.

[6] H. M. Srivastava and S. Owa, Eds., Univalent Functions, Fractional Calculus and their Applications, Ellis Horwood Series: Mathematics and its Applications, Ellis Horwood, Chichester, UK, 1989.

[7] H. M. Srivastava and S. Owa, Eds., Current Topics in Analytic Function Theory, World Sci. Publ. House Company, London, Hong Kong, 1992.

■